

# Conductivity exponents at the percolation threshold

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Connections are found between the two-component percolation problem and the conductor/insulator percolation problem. These produce relations between critical exponents, and suggest formulae connecting the conductivity exponents in different dimensions. Values for the critical exponents are obtained from calculations on the incipient infinite cluster in two and three dimensions.

## I. INTRODUCTION

Percolation is a prototypical example of a critical phenomenon [1]. In particular, a percolating system is characterized by a correlation length  $\xi$  that diverges as the percolation threshold  $p_c$  is approached. At the critical point  $p_c$ , the geometric and dynamic attributes of the infinite, percolating cluster (termed the “incipient cluster”) are described by a set of critical exponents that define a universality class; that is, the set is particular to the (Euclidean) dimension  $d$  of the system. Because fine details of the system are not important, percolation serves as a useful model for natural phenomena [2] where a dynamical process is affected by some sort of constriction. In this paper, however, uncorrelated, isotropic systems are considered, where the interest is in the values of the critical exponents and the relations between them.

Two different approaches to the critical point  $p_c$  are taken by the two-component percolation problem and the more-familiar conductor/insulator percolation problem. These two systems have no geometric attributes in common, but are related by their dynamic exponents.

The two-component percolation problem [3] involves a two-component material system of infinite extent. The higher conductivity phase, having conductivity  $\sigma_1$ , is randomly mixed with the lower conductivity phase ( $\sigma_2$ ); further, the volume fraction  $p$  of the higher conductivity phase is precisely at the percolation threshold  $p_c$ . It is reasonable to expect the effective conductivity  $\sigma$  of the system to exhibit critical behavior as the conductivity value  $\sigma_2$  approaches zero. Indeed, the power-law relation

$$\sigma = \sigma_1 r^u \quad (1)$$

is found to hold for the 2D square lattice over  $0 < r < 1$ , where ratio  $r \equiv \sigma_2/\sigma_1$ .

The conductor/insulator percolation problem involves an insulator phase randomly mixed with a conducting phase of volume fraction  $p > p_c$ . The effective conductivity  $\sigma$  exhibits the *asymptotic* behavior

$$\sigma \sim (p - p_c)^t \quad (2)$$

as  $p$  approaches  $p_c$  from above.

[Some comments on notation: The symbol  $\sim$  (“asymptotically equal to”) is not to be confused with  $\propto$ , which means “proportional to”. The letter  $t$  is used both for the conductivity exponent (as in the equation above) and for the variable “time”; it should be clear from the context, and placement, what meaning should be assumed for  $t$ . In parts of this paper it is convenient to denote an effective conductivity (or resistivity) in a more particular way than is done above. For example,  $\sigma(p, \sigma_1; (1-p), \sigma_2)$  is the effective conductivity of an uncorrelated system comprised of volume fraction  $p$  of sites having conductivity  $\sigma_1$ , and volume fraction  $(1-p)$  of sites having conductivity  $\sigma_2$ .]

The following section presents the Walker Diffusion Method by which many of the analytical and numerical results in this paper are obtained. Subsequent sections are devoted to the two-component percolation problem, the conductor/insulator problem, and numerical methods and results.

## II. WALKER DIFFUSION METHOD

The WDM was developed to calculate effective transport coefficients (e.g., conductivity) of composite materials and systems [4, 5]. This method exploits the isomorphism between the transport equations and the diffusion equation for a collection of non-interacting walkers (hence the name). The phase domains in a composite correspond to distinct populations of walkers, where the walker density of a population is given by the value of the transport coefficient of the corresponding phase domain. The principle of detailed balance ensures that the population densities are maintained, and provides the following rule for walker diffusion over a digitized (pixelated) composite: a walker at site (or pixel)  $i$  attempts a move to a randomly chosen adjacent site  $j$  during the time interval  $\tau = (4d)^{-1}$ , where  $d$  is the Euclidean dimension of the space; this move is successful with probability  $p_{ij} = \sigma_j/(\sigma_i + \sigma_j)$ , where  $\sigma_i$  and  $\sigma_j$  are the transport coefficients for the phases comprising sites  $i$  and  $j$ , respectively. (In practice, the unsuccessful moves inherent in this rule are eliminated by use of the variable residence time algorithm [4].) The path of a walker thus reflects the composition (population density) and morphology of the domains that are encountered, and may be described by

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a diffusion coefficient  $D_w$  that is related to the effective transport coefficient  $\sigma$  by

$$\sigma = \langle \sigma(\mathbf{r}) \rangle D_w \quad (3)$$

where  $\langle \sigma(\mathbf{r}) \rangle$  is the volume average of the constituent transport coefficients. The diffusion coefficient  $D_w$  is calculated from the equation

$$D_w = \frac{\langle R(t)^2 \rangle}{2dt} \quad (4)$$

where the set  $\{R\}$  of walker displacements, each occurring over the time interval  $t$ , comprises a Gaussian distribution that must necessarily be centered well beyond  $\xi$ . (For practical purposes, the correlation length  $\xi$  is the length scale above which the “effective”, or macroscopic, value of a transport property is obtained.)

For displacements  $R < \xi$ , the walker diffusion is anomalous rather than Gaussian due to the heterogeneity of the composite at length scales less than  $\xi$ . There is, however, an additional characteristic length  $\xi_0 < \xi$  below which the composite is effectively homogeneous [6]; this may correspond, for example, to the average phase domain size. A walker displacement of  $\xi$  requiring a travel time  $t_\xi = \xi^2/(2dD_w)$  is then comprised of  $(\xi/\xi_0)^{d_w}$  segments of length  $\xi_0$ , each requiring a travel time of  $t_0 = \xi_0^2/(2dD_0)$ , where  $D_0$  is the walker diffusion coefficient calculated from displacements  $R < \xi_0$ . Setting  $t_\xi = (\xi/\xi_0)^{d_w} t_0$  gives the relation

$$D_w = D_0 \left( \frac{\xi}{\xi_0} \right)^{2-d_w} = \left( \frac{\xi_0^{d_w}}{2dt_0} \right) \xi^{2-d_w} \quad (5)$$

between the walker diffusion coefficient  $D_w$ , the fractal dimension  $d_w$  of the walker path, and the correlation length  $\xi$ .

### III. TWO-COMPONENT PERCOLATION PROBLEM

From the point of view of the WDM, the two-component percolation problem differs from the conductor/insulator percolation problem mainly by the fact that walkers are never “stranded” on finite clusters of conductor sites (until precisely  $r = 0$ ). Thus the approach to the endpoint, which in both cases is percolation only via the incipient cluster, is different and so introduces a different set of critical exponents.

Combining Eqs. (1), (3) and (5) gives the relation

$$r^u = \frac{\langle \sigma \rangle}{\sigma_1} \left( \frac{\xi_0^{d_w}}{2dt_0} \right) \xi^{2-d_w} \quad (6)$$

which upon rearrangement produces

$$\xi = \left( \frac{\langle \sigma \rangle}{\sigma_1} \right)^{1/(d_w-2)} \left( \frac{\xi_0^{d_w}}{2dt_0} \right)^{1/(d_w-2)} r^{-u/(d_w-2)}. \quad (7)$$

Thus the correlation length  $\xi$  diverges as

$$\xi \sim r^{-u/(d_w^*-2)} \quad (8)$$

near  $r = 0$ . The exponent  $d_w^*$  is the limit of the walker path dimension  $d_w$  at  $r = 0$ . Surprisingly, it appears again in the presentation of the conductor/insulator percolation problem, where its numerical value can be ascertained.

A constraint on the value of the conductivity exponent  $u$  arises from the fact that walkers move according to rules based on ratios of conductivities, not absolute values, and thus  $D_w$  is a function of those ratios. This is embodied in the relationship

$$\begin{aligned} \sigma = \langle \sigma(\mathbf{r}) \rangle D_w &= \sigma_1 \left[ p_c + \frac{\sigma_2}{\sigma_1} (1 - p_c) \right] D_w \\ &= \sigma_2 \left[ \frac{\sigma_1}{\sigma_2} p_c + (1 - p_c) \right] D_w \end{aligned} \quad (9)$$

which simplifies to

$$\sigma(p_c, 1; (1 - p_c), r) = r \sigma(p_c, r^{-1}; (1 - p_c), 1). \quad (10)$$

The conductivity  $\sigma$  on the right-hand side of this equation is that for a two-component system where the percolating component has conductivity  $r^{-1} > 1$  while the non-percolating component has fixed conductivity 1. Note that this conductivity diverges as  $r \rightarrow 0$ . Indeed the equation above produces

$$\sigma(p_c, r^{-1}; (1 - p_c), 1) = r^{u-1} \quad (11)$$

where the exponent  $u - 1$  is necessarily less than zero for all dimensions  $d$ . In fact this result proves  $u_d < 1$ .

The exact value of exponent  $u_2$  is obtained in the following way. Note that two random, isotropic systems  $(p, a; q, b)$  and  $(p, a^{-1}; q, b^{-1})^*$  [the presence or absence of the asterisk identifies the system] are *dual* if the resistivity  $\eta(p, a; q, b)$  equals the conductivity  $\sigma(p, a^{-1}; q, b^{-1})^*$  [likewise  $\eta(p, a^{-1}; q, b^{-1})^* = \sigma(p, a; q, b)$ ]. Assuming duality,

$$\begin{aligned} \sigma(p_c, 1; (1 - p_c), r) &= r \sigma(p_c, r^{-1}; (1 - p_c), 1) \\ &= \frac{r}{\eta(p_c, r^{-1}; (1 - p_c), 1)} = \frac{r}{\sigma(p_c, r; (1 - p_c), 1)^*} \end{aligned} \quad (12)$$

showing that

$$\sigma(p_c, 1; (1 - p_c), r) \sigma(p_c, r; (1 - p_c), 1)^* = r. \quad (13)$$

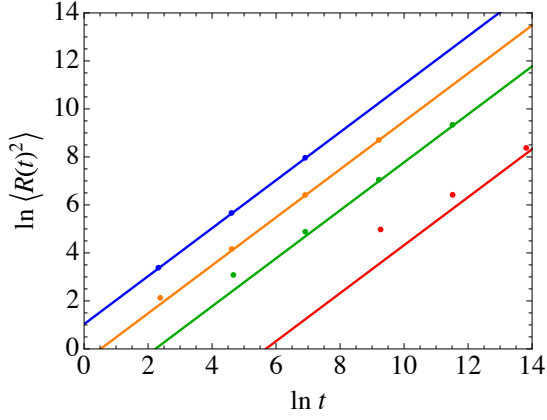


FIG. 1. Data supporting the conjecture that the 3D conductivity exponent  $u_3$  for the two-component percolation problem equals  $3/4$ . The values  $\langle R(t)^2 \rangle$ , each an average over a different set of  $10^5$  two-component systems, are obtained by the WDM; the points would lie on the straight lines (corresponding to  $r = 0.1, 10^{-2}, 10^{-3}, 10^{-5}$ , in order from left to right) in the event that  $u_3 = 3/4$ . The points that lie above the straight lines are affected by their proximity to the correlation length  $\xi$  (which increases with decreasing  $r$ ).

Due to universality, the bond and site implementations of the two-component percolation problem possess the same set of critical exponents  $\{u_d\}$ . Fortuitously, the two-dimensional square bond network is self-dual. For this 2D system the percolation threshold  $p_c = 1/2$  so that

$$\sigma(1/2, 1; 1/2, r) \sigma(1/2, r; 1/2, 1) = r \quad (14)$$

which shows  $\sigma(1/2, 1; 1/2, r) = r^{1/2}$ , meaning  $u_2 = 1/2$ . (A similar argument, using the self-duality of 2D square bond networks, gives the relation  $t_2 - s_2 = 0$  between critical exponents of the conductor/insulator percolation problem. That the argument doesn't provide a value for these exponents suggests that they can only be found asymptotically.)

A numerical value for the exponent  $u_3$  was obtained by the WDM (details of this sort of calculation are given in Sec. V). Figure 1 shows calculated points  $(\ln t, \ln \langle R(t)^2 \rangle)$  for two-component systems with  $r = 0.1, 10^{-2}, 10^{-3}, 10^{-5}$ . The four straight lines represent the relation  $D_w = \sigma / \langle \sigma \rangle$  and so correspond to equations

$$y = x + \ln \left[ \frac{2dr^u}{p_c + (1 - p_c)r} \right] \quad (15)$$

for the four values of  $r$ , with the exponent  $u_3$  set to the value  $0.75$ . The coincidence of the points and the lines support a previous conjecture [7] that  $u_3 = 3/4$ .

The analytical results  $u_d < 1$  and  $u_2 = 1/2$  together with the conjectured result  $u_3 = 3/4$  suggest the relations  $u_{d+1} = (u_d + 1)/2$  and

$$u_d = 1 - (1 - u_2)^{d-1} \quad (16)$$

between the conductivity exponents of the two-component percolation problem.

#### IV. CONDUCTOR/INSULATOR PERCOLATION PROBLEM

The conductor/insulator system has effective conductivity  $\sigma = \sigma_1 p D_w$  where  $p$  is the fraction of conductor sites. As the walker diffusion coefficient  $D_w = \langle R(t)^2 \rangle / (2dt)$  with walk time  $t \gg t_\xi$  is obtained from walkers on *all* conductor sites, not just those on the percolating cluster, the conductor/insulator problem is recast as a two-component problem. Namely, the insulator sites become conductor sites with very low conductivity value  $\sigma_2 \ll \sigma_1$ . Then the conductivity exponent  $t$  is obtained in the limit  $\sigma_2 = 0$  (that is,  $r = 0$ ) at  $p = p_c$ . Thus the correlation length for this two-component system is

$$\xi = \left( \frac{\xi_0^{d_w}}{2dt_0} \right)^{1/(d_w-2)} D_w^{-1/(d_w-2)} \sim p^{1/(d_w^\dagger-2)} (p - p_c)^{-t/(d_w^\dagger-2)}. \quad (17)$$

It is also the case that  $\sigma = \sigma_1 p' D'_w$  where  $p'$  is the fraction of system sites comprising the percolating cluster, and  $D'_w$  is the diffusion coefficient for walkers on the percolating cluster. Additionally, it is known that  $p' \sim (p - p_c)^\beta$  for  $p > p_c$ . Thus

$$\xi \sim (p - p_c)^{\beta/(d_w^*-2)} (p - p_c)^{-t/(d_w^*-2)} = (p - p_c)^{-\nu}. \quad (18)$$

Here the exponent relation  $\nu = (t - \beta)/(d_w^* - 2)$  is obtained, where

$$d_w^* = 2 + (t - \beta)/\nu \quad (19)$$

is the limit of the walker path dimension  $d_w$  at  $p = p_c$ . (Thus  $d_w^*$  is the fractal dimension of the walker path on the incipient infinite cluster.) Note that the walker path dimensions  $d_w^*$  and  $d_w^\dagger$  are related by  $d_w^\dagger - d_w^* = \beta/\nu$ , and that  $d_w^\dagger = 2 + t/\nu$ .

[A more succinct derivation of the exponent relation Eq. (19) is  $\sigma(\xi) = \sigma_1 \rho'(\xi) D'_w(\xi)$  implies  $\xi^{-t/\nu} \propto \xi^{-\beta/\nu} \xi^{2-d_w^*}$ .]

The exponents pertaining to the incipient cluster are additionally connected by a hyperscaling law (a relation that includes the dimension  $d$  of the system). This follows from the asymptotic relation  $p' \sim \xi^{-\beta/\nu}$  and the observation that

$$p' \sim \frac{\xi^D}{\xi^d} \quad (20)$$

where the right-hand side is the volume fraction occupied by the incipient cluster, the exponent  $D$  being the fractal “mass dimension” of that cluster. Thus  $\beta = -\nu(D - d)$  at the percolation threshold.

The appearance of the exponent  $d_w^\dagger$  in both the two-component percolation problem and the conductor/insulator percolation problem can be exploited to find other connections between the systems. Very near the percolation threshold, the effective conductivity of the conductor/insulator system exhibits critical behavior according to the equation

$$\sigma(p > p_c, 1; (1 - p), 0) \sim (p - p_c)^t \sim \xi^{-t/\nu} \quad (21)$$

while the effective conductivity of the conductor/superconductor system exhibits critical behavior described by

$$\sigma(p < p_c, \infty; (1 - p), 1) \sim |p - p_c|^{-s} \sim \xi^{s/\nu}. \quad (22)$$

The exponents  $t$  and  $s$  can be related to  $u$  and  $u - 1$  from the two-component percolation problem by noting that the conductivities of the two conducting systems ( $p > p_c, 1; (1 - p), 0$ ) and ( $p_c, 1; (1 - p_c), r > 0$ ) are identical, and the conductivities of the two superconducting systems ( $p < p_c, \infty; (1 - p), 1$ ) and ( $p_c, r^{-1}; (1 - p_c), 1$ ) are identical, when the parameters  $p$  and  $r$  are very close to  $p_c$  and 0, respectively. For example,

$$\sigma(p > p_c, 1; (1 - p), 0) \sim (p - p_c)^t \sim \xi^{-(d_w^\dagger - 2)} \quad (23)$$

and

$$\sigma(p_c, 1; (1 - p_c), r > 0) = r^u \sim \xi^{-(d_w^\dagger - 2)} \quad (24)$$

so that in both cases  $\sigma(\xi) \propto \xi^{-t/\nu}$ .

The critical behavior of the conductor/superconductor system [Eq. (22)] is also described by

$$\sigma(p < p_c, \infty; (1 - p), 1) \sim |p - p_c|^{-s} \sim \xi^{-(d_w^\dagger - 2)} \quad (25)$$

where the exponent  $d_w^\dagger = 2 - s/\nu$  is the limit of the walker path dimension  $d_w$  for the conductor/superconductor system at  $p = p_c$ . Note that  $d_w^\dagger < 2$ , so the displacement of walkers over time is superdiffusive. Similarly,

$$\sigma(p_c, r^{-1}; (1 - p_c), 1) = r^{u-1} \sim \xi^{-(d_w^\dagger - 2)} \quad (26)$$

where the same exponent  $d_w^\dagger$  is the limit of the walker path dimension at  $r = 0$ . Thus in both cases  $\sigma(\xi) \propto \xi^{s/\nu}$ .

The asymptotic relations  $(p - p_c)^t \sim \xi^{-t/\nu}$  and  $|p - p_c|^{-s} \sim \xi^{s/\nu}$  have counterparts in  $r^u \sim \xi^{-t/\nu}$  and  $r^{u-1} \sim \xi^{s/\nu}$ . These produce  $\xi \sim |p - p_c|^{-\nu}$ , and  $\xi \sim r^{-u(\nu/t)}$  and  $\xi \sim r^{(u-1)(\nu/s)}$  from which it follows that  $r^{-u(\nu/t)} \sim r^{(u-1)(\nu/s)}$ . Thus the conductivity exponents are related as

$$\frac{-u}{t} = \frac{u-1}{s} \quad (27)$$

or equivalently as  $u = t/(s + t)$ , in all dimensions.

A consequence of Eq. (16) is then

$$\frac{t_d}{s_d} = 2^{d-1} - 1. \quad (28)$$

Using the value for exponent  $t_3$  calculated in the following section, the value  $s_3 = 0.67787(105)$  is a prediction.

It is interesting to consider a counterpart to Eq. (16) for the conductor/insulator system. In this case the conductivity exponent  $t_d$  increases towards 3 as the dimension increases [1]. Then

$$t_d = 3 \left[ 1 - \left( 1 - \frac{t_2}{3} \right)^{d-1} \right]. \quad (29)$$

Given the generally accepted value  $t_2 = 1.30$  (1.299), this equation produces  $t_3 = 2.03667$  (2.03553) and similarly reasonable values for higher dimensions.

The diffusivity counterpart to the asymptotic relation  $\sigma \sim (p - p_c)^t$  for conductivity is  $D'_w \sim (p - p_c)^{t-\beta}$ . Then

$$t_d - \beta_d \approx 1.91 \left[ 1 - \left( 1 - \frac{t_2 - \beta_2}{1.91} \right)^{d-1} \right] \quad (30)$$

which also produces very reasonable results. Note that the factor 1.91 is not optimized to give best results. Perhaps these relations connecting dynamic exponents across dimensions indicate a broader concept of “universality”.

## V. NUMERICAL APPROACH AND RESULTS

Because the critical exponents are obtained from the incipient infinite cluster, it is important to ensure that the diffusing walkers, which perform the calculations, are indeed on that cluster. To start, a walker is placed on a conductor site at the center of a vast volume of “undefined” sites. Then each neighboring site is defined to be conducting (with probability  $p_c$ ) or is otherwise insulating. Rather than have the walker then attempt a move to a randomly chosen neighboring site (which may not be successful), it is more efficient to utilize the variable residence time algorithm, which takes advantage of the statistical nature of the diffusion process.

According to this algorithm [4], the actual behavior of the walker is well approximated by a sequence of moves in which the direction of the move from a site  $i$  is determined randomly by the set of probabilities  $\{P_{ij}\}$ , where  $P_{ij}$  is the probability that the move is to adjacent site  $j$  (which has conductivity  $\sigma_j$ ) and is given by the equation

$$P_{ij} = \frac{\sigma_j}{\sigma_i + \sigma_j} \left[ \sum_{k=1}^{2d} \left( \frac{\sigma_k}{\sigma_i + \sigma_k} \right) \right]^{-1}. \quad (31)$$

The sum is over all sites adjacent to site  $i$ . The time interval over which the move occurs is

$$T_i = \left[ 2 \sum_{k=1}^{2d} \left( \frac{\sigma_k}{\sigma_i + \sigma_k} \right) \right]^{-1}. \quad (32)$$

Note that this version of the variable residence time algorithm is intended for orthogonal systems (meaning a site in a 3D system has six neighbors, for example).

After each move, any “undefined” neighboring sites are converted to conducting or insulating. In this way the cluster grows. A walk is complete when the sum of move times  $T_i$  reaches or exceeds a preset walk time  $T$ .

Of course, many of those clusters turn out to be finite and so clearly are not part of the incipient cluster. Indeed, the larger the preset walk time  $T$ , the greater the likelihood that a nascent cluster will turn out to be finite. Finite clusters are identified by the fact that *all* conductor sites comprising the cluster have been visited by time  $T$  (so the cluster is completely surrounded by insulator sites). An “infinite” (or percolating) cluster includes at least one conductor site on the boundary that was “created” by the walker (in the manner described above) but never actually visited in time  $T$ .

In general,  $n \times 10^5$  “infinite” clusters for *each* walk time  $T$  were used to determine the value of a critical exponent or a ratio of exponents. These represent  $n \times 10^5$  *different pieces*, each of size corresponding to the walk time  $T$ , of the incipient cluster. It doesn’t matter that a cluster still “infinite” at time  $T$  might turn out to be finite were the walk extended to longer times, since every finite cluster at the percolation threshold resembles the incipient cluster (which is statistically self-similar over *all* length scales) over length scales up to the size of the cluster.

The numerical data recorded for the incipient cluster was, for each of several preset walk times  $T$ , the following: (1) The number  $N_{pc} = 10^5$  of percolating (“infinite”) clusters over which most other quantities are averaged. (2) The number  $N_{fc}$  of finite clusters encountered in the process of accumulating  $N_{pc}$  percolating clusters. (3) The actual (averaged) walk time  $t$  (very slightly larger than  $T$ ). (4) The average walker displacement  $\langle R(t) \rangle$ . (5) The average walker displacement-squared  $\langle R(t)^2 \rangle$ . (6) The average number  $\langle n_m(t) \rangle$  of walker moves. (7) The average number  $\langle n_s(t) \rangle = \langle S(t) \rangle$  of visited sites.

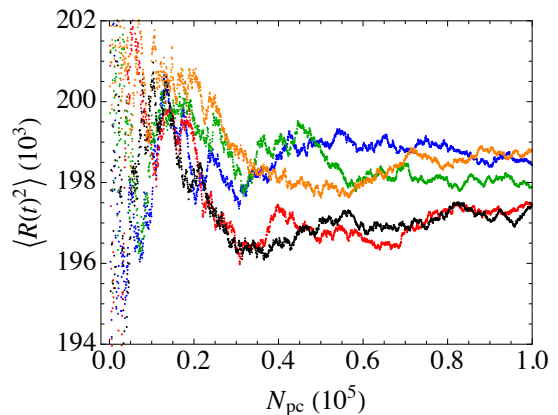


FIG. 2. Sets of points (distinguished by color) that converge toward a “correct” value for the average walker displacement-squared  $\langle R(t)^2 \rangle$  for walks of time  $t = 10^7$  over the incipient cluster in 2D. The variable  $N_{pc}$  corresponds to the number of independent walks from which the average value  $\langle R(t)^2 \rangle$  is obtained.

The percolation threshold values used in the calculations are  $p_c = 0.592746$  (2D) and  $p_c = 0.311607$  (3D). The “standard” values for  $\beta$  and  $\nu$  and  $D$  referred to below are  $\beta_2 = 5/36$  and  $\nu_2 = 4/3$  and  $D_2 = 91/48$  [1]; and  $\beta_3 = 0.41810(57)$  and  $\nu_3 = 0.87642(115)$  and  $D_3 = 2.52295(15)$ , derived from values  $1/\nu_3 = 1.1410(15)$  and  $\beta_3/\nu_3 = 0.47705(15)$  [8].

#### A. Comment on average value $\langle R(t)^2 \rangle$

Most calculations of interest require arguably correct (as well as accurate) values for the average walker displacement-squared  $\langle R(t)^2 \rangle$ . In particular it is important that a sufficient number of independent walks (i.e., walks over a sufficient number of distinct sections of a percolating system) be taken in order that a mean value for  $\langle R(t)^2 \rangle$  with reasonably narrow bounds is obtained. Figures 2 and 3 are instructive on this point.

Figure 2 shows five sets of points (distinguished by color) pertaining to walker diffusion on the incipient infinite cluster in 2D. Consider *one* of those sets: The coordinates of the points are  $(N_{pc}, \langle R(t)^2 \rangle)$ , where the average value  $\langle R(t)^2 \rangle$  is obtained from  $N_{pc}$  percolating clusters (that is, from  $N_{pc}$  independent walks). As more walks are taken (i.e., as  $N_{pc}$  increases), the average value  $\langle R(t)^2 \rangle$  fluctuates less and flattens out. Then by creating *several* sets and reproducing this behavior, a set size  $N_{pc}$  is found ( $10^5$  in this case) that permits a mean value  $\langle \langle R(t)^2 \rangle \rangle$  to be obtained with reasonably narrow bounds.

Similarly, Fig. 3 shows five sets of points pertaining to walker diffusion on the incipient cluster in 3D. Again, sets of size  $N_{pc} = 10^5$  appear to be sufficient to obtain a defensible value for  $\langle R(t)^2 \rangle$  for use in calculations. (Larger sets may naturally reduce the bounds, but at the cost of significantly increased computer time.)



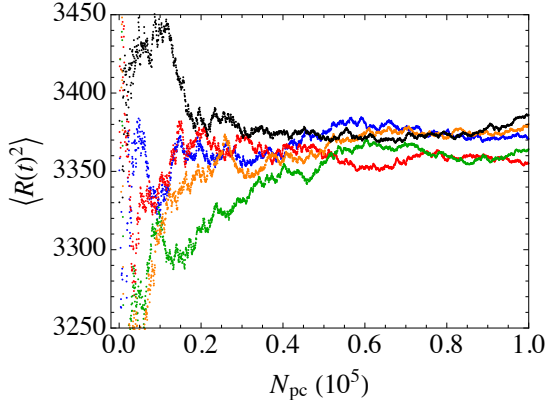


FIG. 3. Sets of points (distinguished by color) that converge toward a “correct” value for the average walker displacement-squared  $\langle R(t)^2 \rangle$  for walks of time  $t = 10^6$  over the incipient cluster in 3D. The variable  $N_{pc}$  corresponds to the number of independent walks from which the average value  $\langle R(t)^2 \rangle$  is obtained.

Data from Figs. 2 (walk time  $t = 10^7$ ) and 3 ( $t = 10^6$ ) are used (together with additional sets of size  $10^5$ ) in the calculations of  $d_w^*$  below.

### B. Walker path dimension $d_w^*$

For percolating systems of size  $L < \xi$ , the equivalent of Eq. (5) is

$$D_w(L) = D_w(\xi) \left( \frac{L}{\xi} \right)^{2-d_w} = \left( \frac{\xi_0^{d_w}}{2dt_0} \right) L^{2-d_w}. \quad (33)$$

In the case of the incipient cluster, which is statistically self-similar over *all* length scales, this relation can be expressed in terms of the computable variable  $\langle R(t)^2 \rangle$ , namely,

$$\begin{aligned} \frac{\langle R(t)^2 \rangle}{2dt} &= \left( \frac{\xi_0^{d_w}}{2dt_0} \right) \langle R(t)^2 \rangle^{1-d_w^*/2} \\ &= \langle R(t)^2 \rangle^{1-d_w^*/2}. \end{aligned} \quad (34)$$

The last equality comes about because the correlation length  $\xi_0$  is the size of a single conductor site; that is,  $\xi_0 = 1$ . This Gaussian regime corresponds to walkers diffusing within the conductor site for walk times  $t < t_0$ . Then the diffusion coefficient  $D_0 = 1$  and so the travel time  $t_0 = (2d)^{-1}$ . Thus

$$\langle R(t)^2 \rangle = (2dt)^{2/d_w^*} \quad (35)$$

or equivalently,

$$\ln \langle R(t)^2 \rangle = \frac{2}{d_w^*} \ln t + \frac{2}{d_w^*} \ln(2d). \quad (36)$$

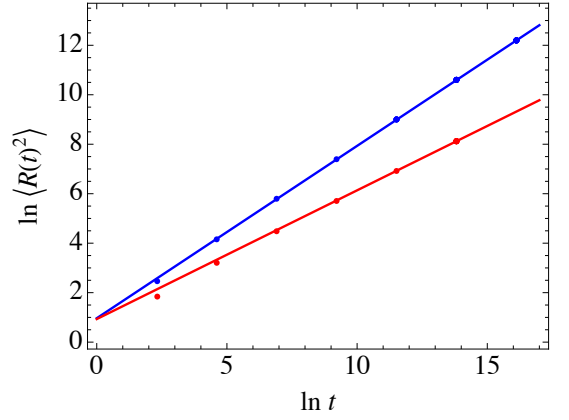


FIG. 4. Data obtained from walks over the 2D and 3D incipient clusters by which the walker path dimension  $d_w^*$  is calculated. Both straight lines have slope  $2/d_w^*$ ; the line of greater (lesser) slope corresponds to the 2D (3D) incipient cluster. The value  $\langle R(t)^2 \rangle$  for each point is obtained from one or more sets of  $10^5$  distinct sections of the incipient cluster. Points at short walk times  $t$  are affected by the finite size of the conductor sites.

This last equation produces the straight lines in Fig. 4. The line of greater (lesser) slope, running through the point corresponding to largest walk time  $t$ , has slope inversely proportional to the walker path dimension  $d_w^*$  for 2D (3D) percolation. Note that in both cases, points corresponding to shorter walk times lie below the straight lines, due to the effect of the finite (not infinitesimal) size of the conductor sites. [A more precise explanation is as follows: Walker diffusion on the incipient cluster comprised of conductor sites is Gaussian ( $d_w = 2$ ) for walk times  $t < t_0$ , and anomalous ( $d_w = d_w^* > 2$ ) for walk times  $t > t_0$ , so that lines of slope 1 and slope  $2/d_w^*$  meet at the point  $(\ln t_0, \ln \langle R(t_0)^2 \rangle) = (-\ln 2d, \ln 1)$ . Points in the anomalous regime near  $t = t_0$  are thus affected by the presence of the Gaussian regime and so lie *below* the slope  $2/d_w^*$  line. Note that this behavior contrasts with that exhibited by bond systems at  $p_c$ . In the bond case the walkers reside at the zero-dimensional nodes [5], so there is no Gaussian regime near  $t = 0$ . Thus all points lie on the slope  $2/d_w^*$  line.]

For 2D percolation, the value  $d_w^* = 2.87038(60)$  was obtained from  $10^6$  walks, each of duration  $T = 10^7$ , over ten sets of  $10^5$  clusters (representing  $10^6$  distinct sections of the incipient infinite cluster). The average number of moves per walk  $\langle n_m \rangle > 25 \times 10^6$ , and the average number of visited sites per walk  $\langle n_s \rangle > 68 \times 10^3$ .

For 3D percolation, the value  $d_w^* = 3.84331(193)$  was obtained from  $8 \times 10^5$  walks, each of duration  $T = 10^6$ , over eight sets of  $10^5$  clusters (representing  $8 \times 10^5$  distinct sections of the incipient infinite cluster). The average number of moves per walk  $\langle n_m \rangle > 2.3 \times 10^6$ , and the average number of visited sites per walk  $\langle n_s \rangle > 12 \times 10^3$ .

In both cases Fig. 4 shows that these walks are of sufficient length (sufficient walk time) that finite-site-size effects on these  $d_w^*$  values are negligible, and Figs. 2 and

3 show that a sufficient number of randomly selected sections of the incipient cluster are explored to give exponent values within meaningful brackets.

With these values for  $d_w^*$  and the standard values for  $\beta$ ,  $\nu$ , and  $D$ , the exponent relations from the previous section produce the following values for the exponent ratio  $t/\nu$ , the conductivity exponent  $t$ , and the spectral dimension  $d_s = 2D/d_w^*$ :

	$t/\nu$	$t$	$d_s$
2D	0.974542(600)	1.29939(80)	1.32097(28)
3D	2.32036(193)	2.0336(32)	1.3129(7)

Note that the data from these  $n \times 10^5$  walks over time  $T = 10^7$  (2D) or  $10^6$  (3D) are used in all the following calculations that pertain to the incipient infinite cluster. Data for shorter walk times  $T = 10, 10^2, 10^3, \dots$  are obtained from one or more sets of  $10^5$  walks.

### C. Incipient cluster mass dimension $D$

A lower bound  $D_s$  on the mass dimension  $D$  of the incipient cluster is found by considering the number  $S(t)$  of distinct sites visited during a walk to be proportional to  $R_c(t)^{D_s}$ , where  $R_c(t)$  is the crude radius of the cluster of visited sites. This cluster radius can be related to the walker displacement  $R(t)$  by noting that the walker is essentially equilibrated after many moves over the cluster of visited sites ( $n_m/n_s \gg 1$ ). Then the displacement  $R(t)$  finds the walker at any site of the cluster with equal probability. For example, in the case of a walker confined to a 3D spherical cluster of conductor sites, the average value  $\langle r \rangle$  is given by

$$\langle r \rangle = \left( \frac{4}{3} \pi R_c^3 \right)^{-1} \int_0^{R_c} r \cdot 4\pi r^2 dr = \frac{3}{4} R_c \quad (37)$$

since  $r$ , that is  $R(t)$ , is measured from the origin of the cluster (the original site from which the cluster grew). More generally,  $R_c \propto \langle R(t) \rangle$  and therefore

$$\langle S(t) \rangle \propto \langle R(t) \rangle^{D_s} \quad (38)$$

with the averages obtained from a very large number of clusters and walks.

This relation produces the straight lines in Figs. 5 and 6 which describe the growth of the cluster of visited sites produced by walkers confined to the incipient cluster. In Fig. 5 the slope  $D_s = 1.89503$  is obtained for 2D percolation; this  $D_s$  value is slightly less than the fractal dimension  $D = 91/48 = 1.89583$  of the incipient cluster. In Fig. 6 the slope  $D_s = 2.49848$  is obtained for 3D percolation; similarly, this  $D_s$  value is slightly less than the standard value  $D = 2.52295(15)$  for the incipient cluster

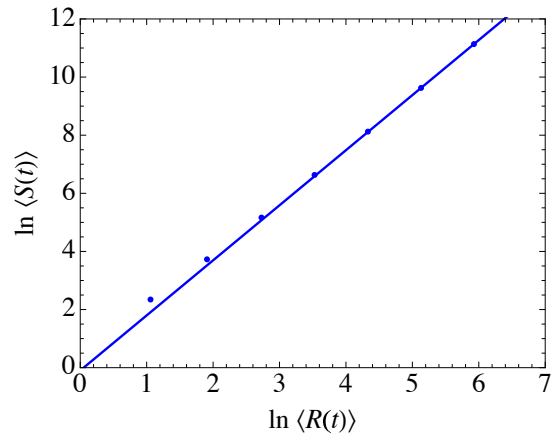


FIG. 5. Data obtained from walks over the 2D incipient cluster by which the fractal dimension  $D_s$  of the cluster  $S(t)$  of visited sites is calculated. The straight line fit to points for  $t = 10^6$  and  $10^7$  has slope  $D_s$ , giving a lower bound for the fractal dimension  $D$  of the incipient cluster. The values  $\langle R(t) \rangle$  and  $\langle S(t) \rangle$  for each point are obtained from one or more different sets of  $10^5$  distinct sections of the incipient cluster. Points at short walk times  $t$  are affected by the finite size of the conductor sites.

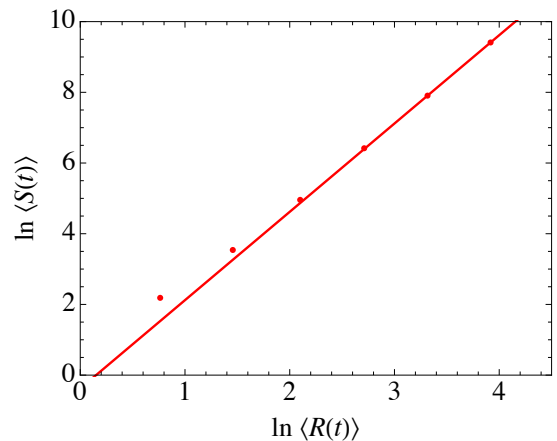


FIG. 6. Data obtained from walks over the 3D incipient cluster by which the fractal dimension  $D_s$  of the cluster  $S(t)$  of visited sites is calculated. The straight line fit to points for  $t = 10^5$  and  $10^6$  has slope  $D_s$ , giving a lower bound for the fractal dimension  $D$  of the incipient cluster. The values  $\langle R(t) \rangle$  and  $\langle S(t) \rangle$  for each point are obtained from one or more different sets of  $10^5$  distinct sections of the incipient cluster. Points at short walk times  $t$  are affected by the finite size of the conductor sites.

[8]. In both cases the line was fit to the two largest-walk-time points (each point obtained from eight or more sets of  $10^5$  independent walks) in order to minimize the effects of the finite (not infinitesimal) size of the conductor sites apparent at shorter times  $t$ .

While the value  $D_s$  may be very close to  $D$ , it will always be smaller since the cluster  $S(t)$  will never completely fill the section of the incipient cluster explored by the walker over time  $t$  (the walker will never visit ev-

ery accessible site in that section). An extreme example of this effect is walker diffusion over a homogeneous 2D system: the path dimension  $d_w$  is found to be precisely 2, but  $D_s \approx 1.885$  (far less than  $D = d = 2$ ) since the cluster  $S(t)$  in that case grows in a non-compact way and so suggests a system with dimension less than 2.

#### D. Fraction $p'$

The fraction  $p'$  of system sites comprising the percolating cluster appears in the expression for conductivity  $\sigma = \sigma_1 p' D'_w$  for systems with  $p > p_c$ , and in the relation  $p' \sim (p - p_c)^\beta$  for infinite systems very close to the percolation threshold. The function  $p'$  is derived here, as it is used in calculations below.

It is reasonable to assume that a created cluster of size greater than the correlation length  $\xi$  (which occurs when the created cluster is “infinite” at preset walk time  $T \gg t_\xi$ ) is part of the infinite percolating cluster. A very large number  $N_{pc}$  of such “infinite” clusters are needed in the calculation of  $D'_w$ . In the process of creating these  $N_{pc}$  percolating clusters, a number  $N_{fc}$  of smaller, “finite” clusters are generated that cannot be used in the calculation of  $D'_w$ . Recall that creation of each cluster ( $N_{pc} + N_{fc}$  in total) begins at a randomly selected conductor site (in practice, at a conductor site completely surrounded by a sea of “undefined” sites). As this random selection of conductor sites will produce a fraction  $p'/p$  that belong to the percolating cluster, the fraction  $N_{pc}/(N_{pc} + N_{fc}) = p'/p$ ; that is,

$$p' = p \left( 1 + \frac{N_{fc}}{N_{pc}} \right)^{-1}. \quad (39)$$

#### E. Exponent ratio $\beta/\nu$

The asymptotic relations  $p' \sim (p - p_c)^\beta$  and  $\xi \sim (p - p_c)^{-\nu}$  produce  $p' \sim \xi^{-\beta/\nu}$ . This inspires a finite-size scaling relation  $p'(L) \propto L^{-\beta/\nu}$  that gives the fraction of sites in an arbitrary portion of size  $L$  of an infinite system at  $p = p_c$ , that belong to the cluster that percolates the size  $L$  volume. The equivalent scaling relation for  $p'(t)$  is  $p'(t) \sim \langle R(t) \rangle^{-\beta/\nu}$  or  $p'(t) \sim \langle R(t)^2 \rangle^{-\beta/2\nu}$ . These are asymptotic relations because the expression  $p'(t) = p_c(1 + N_{fc}/N_{pc})^{-1}$  [Eq. (39) with  $p = p_c$ ] is valid only for systems of size larger than the correlation length  $\xi$ .

The first formulation produces the straight lines (with slope approximating  $-\beta/\nu$ ) in Figs. 7 and 8. In both the 2D (Fig. 7) and 3D (Fig. 8) cases the fits are to the points for the two largest walk times (each point obtained from eight or more sets of  $10^5$  independent walks). These produce values  $\beta_2/\nu_2 = 0.101027$  (compare to the exact value  $5/48 = 0.104167$  [1]) and  $\beta_3/\nu_3 = 0.454446$  (compare to the value  $0.47705(15)$  [8]). The second formulation gives very similar values:  $\beta_2/\nu_2 = 0.100952$  and  $\beta_3/\nu_3 = 0.453645$ .

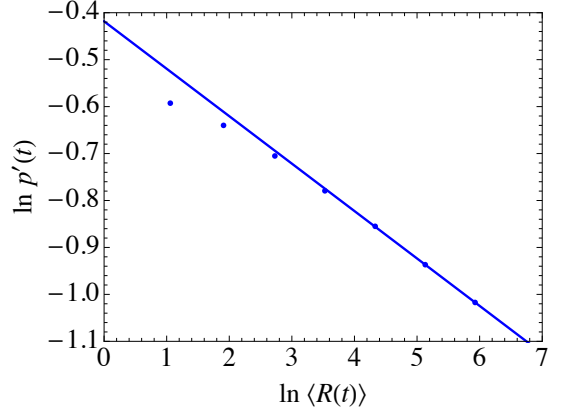


FIG. 7. Data obtained from walks over the 2D incipient cluster by which the exponent ratio  $\beta_2/\nu_2$  is calculated. Values  $\langle R(t) \rangle$  and  $p'(t)$  are obtained for walk times  $t = 10, 10^2, \dots, 10^7$ . The straight line fit to the two points at the largest walk times has slope approximating  $-\beta_2/\nu_2$ .

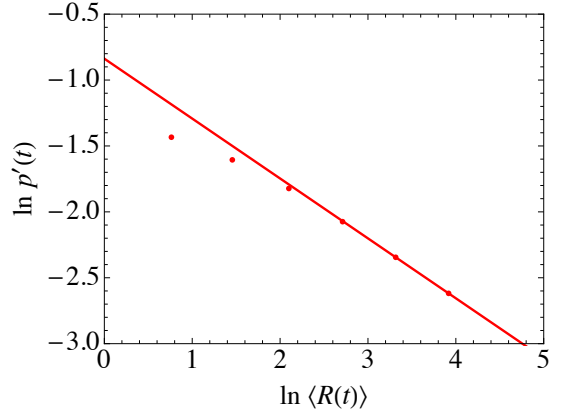


FIG. 8. Data obtained from walks over the 3D incipient cluster by which the exponent ratio  $\beta_3/\nu_3$  is calculated. Values  $\langle R(t) \rangle$  and  $p'(t)$  are obtained for walk times  $t = 10, 10^2, \dots, 10^6$ . The straight line fit to the two points at the largest walk times has slope approximating  $-\beta_3/\nu_3$ .

As points are obtained at ever-larger walk times, the slopes of the fitted lines will increase in magnitude, giving values for the exponent ratio  $\beta/\nu$  closer to the true ones.

#### F. Conductivity $\sigma$ of percolating systems with $p > p_c$

For this case ( $p > p_c$ ) the effective conductivity  $\sigma = \sigma_1 p' D'_w$  where  $p'$  is the fraction of system sites comprising the percolating cluster, and  $D'_w = \langle R(t)^2 \rangle / (2dt)$  is the diffusion coefficient for walkers on the percolating cluster. Walk times  $t$  should be sufficiently large that  $\langle R(t)^2 \rangle \gg \xi^2$ . (Or equivalently, walk times  $t$  should be sufficiently large that  $D'_w$  has reached a constant value. Walk times  $t < t_\xi$  produce  $D'_w$  values that are too high.) The function  $p'(t) = p/(1 + N_{fc}/N_{pc})$  where the ratio



$N_{\text{fc}}/N_{\text{pc}}$  is obtained in the course of generating the large number of walks of time  $t \gg t_\xi$ .

### G. Exponent ratio $s/\nu$

This ratio appears in the conductor/superconductor percolation problem. In particular, the effective conductivity  $\sigma$  of this system diverges as  $\xi^{-(d_w^\dagger-2)}$  where the exponent  $d_w^\dagger = 2 - s/\nu$  is the walker path dimension at  $p = p_c$ . Thus the ratio  $s/\nu$  can be obtained via the relation

$$\langle R(t)^2 \rangle = (2dt)^{2/d_w^\dagger}. \quad (40)$$

It was noted above that  $d_w^\dagger < 2$  indicates the walks needed for this calculation are superdiffusive, meaning some individual moves are very large, in the manner of Lévy flights. This occurs because a walker, upon entering a cluster of superconductor sites, is immediately expelled from a randomly chosen periphery site of that same cluster; in this manner the superconductor cluster acts both as an absorbing boundary and source of walkers [5]. Thus a walk of time  $T$  consists of diffusion through the conductor phase, punctuated by long jumps across superconductor clusters during which no time passes. This ensures that walker populations (reflecting the distribution of conductor and superconductor sites) throughout the system are maintained.

### H. Conductivity exponent $u_3$

For the two-component system, the effective conductivity  $\sigma = \langle \sigma \rangle D_w$  where  $\langle \sigma \rangle = p_c \sigma_1 + (1 - p_c) \sigma_2$ , and  $D_w = \langle R(t)^2 \rangle / (2dt)$  is the walker diffusion coefficient obtained for walk times  $t \gg t_\xi$ . While every system site is accessible to a walker (in contrast to the conductor/insulator system), it is convenient to use the same “created cluster” code.

Thus the walker is placed on a site that is randomly chosen to be of the  $\sigma_1$  sort (with probability  $p_c$ ) or of the  $\sigma_2$  sort (with probability  $1 - p_c$ ). Then each neighboring site is defined to be of the  $\sigma_1$  sort (with probability  $p_c$ ) or of the  $\sigma_2$  sort (with probability  $1 - p_c$ ). Then the walker moves to one of those sites over a time  $T_i$  as dictated by the variable residence time algorithm. And so on.

The 3D results for ratios  $\sigma_2/\sigma_1 = 0.1, 10^{-2}, 10^{-3}, 10^{-5}$  are shown in Fig. 1. As discussed near the end of Sec. III, they support a previous conjecture that  $u_3 = 3/4$ .

## VI. CONCLUDING REMARKS

The intent of this research was to clarify the relationship between the two-component percolation problem and the familiar conductor/insulator percolation problem. The Walker Diffusion Method provided a new conceptual, analytical, and numerical approach to this task.

An important achievement is the introduction of a new critical exponent  $d_w^\dagger$  that connects the two types of percolating systems. This is the fractal dimension of the walker path in the two-component system at the endpoint  $r = 0$ . It is also the limit of the walker path dimension  $d_w$  in the conductor/insulator system when all conductor clusters are connected by an extremely low conductivity “background” (replacing the insulator phase), attained at  $p = p_c$  and background conductivity reduced to zero. The connection made apparent by  $d_w^\dagger$  (and the equivalent exponent  $d_w^\ddagger$  for the superconducting systems) leads to Eq. (27), relating the conductivity exponent  $t$  and superconductivity exponent  $s$ , and the corresponding exponents  $u$  and  $1 - u$ .

The value  $d_w^\dagger$  is best calculated from the exponent relation  $d_w^\dagger = 2 + t/\nu$  derived in Sec. IV. Use of the calculated value for  $t_2$  and the standard value for  $\nu_2$  produce  $d_w^\dagger = 2.97454(60)$  for 2D systems. Alternatively,  $d_w^\dagger$  may be obtained via the relation

$$\langle R(t)^2 \rangle = (2dt)^{2/d_w^\dagger} \quad (41)$$

describing walks over the conductor/insulator system at  $p = p_c$ , where walkers on the finite clusters (in addition to those on the incipient cluster) are included in the calculation. Those trapped walkers diffuse according to the variable residence time algorithm during the walk time  $T$ , and so contribute to the average displacement-squared  $\langle R(t)^2 \rangle$  (hence  $d_w^\dagger > d_w^*$ ).

Additionally, very good values for the critical exponent  $d_w^*$  in two and three dimensions are obtained, which enable calculation of accurate values for the conductivity exponents  $t_2$  and  $t_3$ . WDM calculations also support the conjectured value  $u_3 = 3/4$ , which motivates a proposed set of equations connecting conductivity exponents across dimensions.

## ACKNOWLEDGMENTS

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